COLORED FINITE TYPE INVARIANTS AND A MULTI-VARIABLE ANALOGUE OF THE CONWAY POLYNOMIAL

Sergey A. Melikhov

Abstract. The multi-variable Alexander polynomial (in the form of Conway's potential function), when stripped of a redundant summand, is shown to be of the form $\triangledown_L(x_1-x_1^{-1},\dots,x_m-x_m^{-1})$ for some polynomial $\triangledown_L$ over $\mathbb{Z}$; the Conway polynomial $\nabla_L(z)$ coincides with $z\triangledown_L(z,\dots,z)$. The coefficients of $\triangledown_L$ and of the power series $\triangledown_L^* := \triangledown_L/(\triangledown_{K_1}\cdots\triangledown_{K_m})$, where K_i denote the components of L, are finite type invariants in the sense of Kirk–Livingston. When $m=2$, they are integral liftings of Milnor's invariants $\bar\mu(1\dots12\dots2)$ of even length, including, in the case of $\triangledown_L^*$, Cochran's derived invariants β^k. The coefficients of $\triangledown_L$ and $\triangledown_L^*$ are closely related to certain $\mathbb{Q}$-valued invariants of (genuine) finite type, among which we find alternative extensions $\hat\beta^k$ of β^k to the case $\mathrm{lk}\neq 0$, such that $2\hat\beta^1/\mathrm{lk}^2$ is the Casson–Walker invariant of the $\mathbb{Q}$-homology sphere obtained by 0-surgery on the link components.

Each coefficient of $\triangledown_L^*$ (hence of $\nabla_L^* := \nabla_L/(\nabla_{K_1}\cdots\nabla_{K_m})$) is invariant under TOP isotopy and under sufficiently close C^0-approximation, and can be extended, preserving these properties, to all topological links. The same holds for H_L^* and F_L^*, where H_L and F_L are certain exponential parameterizations of the two-variable HOMFLY and Kauffman polynomials. Next, we show that no difference between PL isotopy and TOP isotopy (as equivalence relations on PL links in S^3) can be detected by finite type invariants. These are corollaries of the fact that any type k invariant (genuine or Kirk–Livingston), well-defined up to PL isotopy, assumes same values on k-quasi-isotopic links. We prove that c_k, where $\nabla_L^*(z) = z^{m-1}(c_0+c_1z^2+c_2z^4+\dots)$, is invariant under k-quasi-isotopy, but c_2 fails to have type 2 in either theory.

1. Introduction

Recall that *PL isotopy* can be viewed as the equivalence relation generated by ambient isotopy and insertion of local knots. The "links modulo knots" version of the Vassiliev Conjecture for knots [Va; Conjecture 6.1] could be as follows.

Problem 1.1. *Are PL isotopy classes of classical links separated by finite type invariants that are well-defined up to PL isotopy?*

Some geometry behind this question is revealed by

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Theorem 1.2. *The answer to 1.1 is affirmative if and only if the following four statements hold simultaneously:*

(i) finite type invariants, well-defined up to PL isotopy, are not weaker than finite KL-type invariants (cf. §2)*, well-defined up to PL isotopy;*

(ii) indistinguishability by finite KL-type invariants, well-defined up to PL isotopy, implies k-quasi-isotopy for all $k \in \mathbb{N}$;

(iii) k-quasi-isotopy for all $k \in \mathbb{N}$ implies TOP isotopy in the sense of Milnor (i.e. homotopy within the class of topological embeddings);

(iv) TOP isotopy implies PL isotopy.

The assertion follows since the converse to each of (i)–(iv) holds with no proviso: to (ii) by Theorem 2.2 below, to (iii) by Theorem 1.3(b) below, to (i) and (iv) by definitions.

We recall that PL links $L, L'\colon mS^1 \hookrightarrow S^3$ (where $mS^1 = S^1_1 \sqcup \dots \sqcup S^1_m$) are called *$k$-quasi-isotopic* [MR1], if they are joined by a generic PL homotopy, all whose singular links are k-quasi-embeddings. A PL map $f\colon mS^1 \to S^3$ with precisely one double point $f(p) = f(q)$ is called a *k-quasi-embedding*, $k \in \mathbb{N}$, if, in addition to the singleton $P_0 = \{f(p)\}$, there exist compact polyhedra $P_1, \dots, P_k \subset S^3$ and arcs $J_0, \dots, J_k \subset mS^1$ such that $f^{-1}(P_j) \subset J_j$ for each $j \le k$ and $P_j \cup f(J_j) \subset P_{j+1}$ for each $j < k$, where the latter inclusion is null-homotopic for each $j < k$.

This definition is discussed in detail and illustrated by examples in [MR1] and [MR2]. The only result we need from these papers is the following obvious

Theorem 1.3. [MR1; Corollary 1.4] *(a) For each $k \in \mathbb{N}$, k-quasi-isotopy classes of all sufficiently close PL approximations to a topological link coincide.*

(b) TOP isotopic PL links are k-quasi-isotopic for all $k \in \mathbb{N}$.

The paper is organized as follows. §2 contains a little bit of general geometric theory of finite KL-type invariants, which we interpret in the context of colored links. (We do not attempt at extending any of the standard algebraic results of the theory of finite type invariants.) The rest of the paper is devoted to specific examples. Except for invariants of genuine finite type — from the Conway polynomial (§3), the multi-variable Alexander polynomial (Theorems 4.2 and 4.3) and the HOMFLY and Kauffman polynomials (§5), — we need to work just to produce interesting examples (Theorems 4.4 and 4.6). The payoff is a further clarification of the relationship between the Alexander polynomial and geometrically more transparent invariants of Milnor [Mi] and Cochran [Co] (Theorem 4.8). We are also able to extract additional geometric information (Theorem 3.4 and Corollary 4.10b), which was the initial motivation for this paper (compare [MR1; Problem 1.5]).

2. Colored finite type invariants and k-quasi-isotopy

We introduce finite type invariants of colored links, as a straightforward generalization of the Kirk–Livingston setting of finite type invariants of links [KL]. Kirk–Livingston type k invariants are recovered as type k invariants of colored links whose components have pairwise distinct colors, whereas the usual type k invariants of links (as defined by Goussarov [Gu1] and Stanford [St]) coincide with type k invariants of monochromatic links. Although normally everything boils down to one of these two extreme cases, the general setting, apart from its unifying appeal, may be useful in proofs (cf. proof of Proposition 3.1) and in dealing with colored

link polynomials (cf. Theorem 4.6). We refer to [Va], [Gu], [PS], [BL], [St], [Ba] for a background on finite type invariants.

An m-component classical *colored link map* corresponding to a given *coloring* $c\colon \{1,\dots,m\}\to\{1,\dots,m\}$ is a continuous map $L\colon mS^1\to S^3$ such that $L(S^1_i)\cap L(S^1_j)=\emptyset$ whenever $c(i)\neq c(j)$. Thus usual link maps correspond to the identity coloring and arbitrary maps to the constant coloring. Let $LM^3_{1,\dots,1}(c)$ be the space of all PL c-link maps $mS^1\to S^3$, and let $\mathcal{LM}(c)$ denote its subspace consisting of maps whose only singularities are transversal double points (the integer m will be fixed throughout this section). Note that $\mathcal{LM}(c)$ is disconnected unless c is constant. Let $\mathcal{LM}_n(c)$ (resp. $\mathcal{LM}_{\geqslant n}(c)$) denote the subspace of $\mathcal{LM}(c)$ consisting of maps with precisely (resp. at least) n singularities. Note that $\mathcal{LM}_0(c)$ does not depend on c.

Given any ambient isotopy invariant $\chi:\mathcal{LM}_0(c)\to G$ with values in a finitely generated abelian group G, it can be extended to $\mathcal{LM}(c)$ inductively by the formula

$$\chi(L_s)=\chi(L_+)-\chi(L_-),$$

where $L_+,L_-\in\mathcal{LM}_n(c)$ differ by a single positive crossing of components of the same color, and $L_s\in\mathcal{LM}_{n+1}(c)$ denotes the intermediate singular link with the additional double point. If thus obtained extension $\bar\chi$ vanishes on $\mathcal{LM}_{\geqslant r+1}(c)$, i.e. on all c-link maps with at least $r+1$ transversal double points, for some finite r, then χ is called a *finite c-type* invariant, namely of *c-type r*. Note that in our notation any c-type r invariant is also of c-type $r+1$. Sometimes we will identify a finite c-type invariant $\chi\colon\mathcal{LM}_0(c)\to G$ with its extension $\bar\chi\colon\mathcal{LM}(c)\to G$.

For $m=1$ there is only one coloring c, and finite c-type invariants, normalized by $\chi(\text{unknot})=0$, coincide with the usual Goussarov–Vassiliev invariants of knots. If c is a constant map, we write 'type' instead of 'c-type', and if c is the identity, we write 'KL-type' instead of 'c-type', and $\mathcal{LM}_*$ instead of $\mathcal{LM}_*(c)$. If the coloring c is a composition $c=de$, then any d-type k invariant is also a c-type k invariant. In particular, any type k invariant is a KL-type k invariant; but not vice versa. Indeed, the linking number $\mathrm{lk}(L)$ and the generalized Sato-Levine invariant (see §3) are of KL-types 0 and 1, respectively, but of types exactly 1 and 3.

It is arguable that triple and higher $\bar\mu$-invariants with distinct indices should be regarded as having KL-type 0 (as they assume values in different cyclic groups depending on link homotopy classes of proper sublinks, we must either agree to imagine these groups as subgroups, say, of $\mathbb{Z}\oplus\mathbb{Q}/\mathbb{Z}$, or extend the definition of finite KL-type invariants to include the possibility where each path-connected component of $\mathcal{LM}$ maps to its own range). In contrast, there are almost no finite type invariants of link homotopy [MT], although this can be remedied by consideration of string links (see references in [Hu] or [MV]) or partially defined finite type invariants [Gu1; Remark 10.2], [Hu; §5]. We show in §4 that certain $\bar\mu$-invariants of 2-component links have integer liftings of finite KL-type, at least some of which are not of finite type due to

Proposition 2.1. *The KL-type 0 invariant $v(L)=(-1)^{\mathrm{lk}(L)}$ is not of finite type.*

Proof. By induction, $v(L_s)=\pm 2^k$ for any singular link L_s with k transversal intersections between distinct components and no self-intersections. □

It is clear that KL-type r invariants $\chi:\mathcal{LM}_0\to G$ form an abelian group, which we denote by G_r (or $G_r^{(m)}$ when the number of components needs to be specified).

Clearly, G_0 is the direct sum of $|\pi_0(\mathcal{LM})|$ copies of G. Let $G_r(\lambda)$ denote the subgroup of G_r consisting of invariants vanishing on all links whose link homotopy class is not $\lambda \in \pi_0(\mathcal{LM})$, and on a fixed link $B_\lambda \in \lambda$ with unknotted components. The latter can always be achieved by adding a KL-type 0 invariant, so the quotient map $q\colon G_r \to G_r/G_0$ takes $G_r(\lambda)$ isomorphically onto a subgroup, independent of the choice of B_λ, moreover $G_r/G_0 = \bigoplus_\lambda q(G_r(\lambda))$. For any $\mathrm{lk} \in \mathbb{Z}$, it was proved in [KL] that $\mathbb{Z}_1^{(2)}(\mathrm{lk}) \simeq \mathbb{Z}$ (generated by the generalized Sato–Levine invariant) and conjectured that $\mathbb{Z}_r^{(2)}(\mathrm{lk})$ is not finitely generated for $r > 1$. It is well-known [St] that the group of monochromatic G-valued type r invariants of m-component links is finitely generated; in particular, so is $G_r^{(1)}$. Let $\tilde G_r$ denote the subgroup of G_r consisting of those invariants which remain unchanged under tying local knots, i.e. under PL isotopy. Notice that the well-known proof that type 1 invariants of knots are trivial works also to show that $\tilde G_1 = G_1$. In the following remarks we address the difference between G_r and $\tilde G_r$ for $r > 1$.

Remarks. (i). By evaluating any type r invariant of knots on the components of a link $L \in \lambda$ we obtain a monomorphism

$$\Lambda\colon \bigoplus_{i=1}^{m} G_r^{(1)} \hookrightarrow G_r(\lambda).$$

We claim that its image (whose elements we regard here as the 'least interesting' among all) is a direct summand, with complement containing $\tilde G_r(\lambda)$. Indeed, for any $\chi \in G_r(\lambda)$ define a knot invariant χ_i by $\chi_i(K) = \chi(K_i)$, where K_i is the link, obtained by tying the knot K locally on the i^{th} component of B_λ. Then χ_i is a type r knot invariant, since the local knot on the i^{th} component of K_i, viewed as a knotted ball pair (B^3, B^1), can be chosen to look precisely the same as K outside a small ball. Define an endomorphism Φ of $G_r(\lambda)$ by $\Phi(\chi)(L) = \chi(L) - \sum_{i=1}^m \chi_i(L^i)$, where L^i denotes the i^{th} component of L. Then Φ takes any $\chi \in \operatorname{im}\Lambda$ to zero and any $\chi \in \tilde G_r(\lambda)$ to itself, and consequently defines a splitting (depending, in general, on the choice of B_λ) of the quotient map $G_r(\lambda) \to \operatorname{coker}\Lambda$, with image containing $\tilde G_r(\lambda)$.

(ii). We claim that an invariant χ in the complement to $\operatorname{im}\Lambda$ is invariant under tying local knots iff the restriction of $\bar\chi$ to $\mathcal{LM}_1$ (which is a KL-type $r-1$ invariant of singular links) is. Indeed, the 'only if' implication is trivial. Conversely, it suffices to find one link $L \in \lambda$ such that $\chi(L)$ is unchanged when any local knot is added to L. Since $\chi = \Phi(\chi)$, clearly B_λ is such a link.

In particular, since KL-type 1 invariants of singular links are invariant under tying local knots due to the one-term (framing independence) relation, it follows that

$$G_2(\lambda) = \tilde G_2(\lambda) \oplus \operatorname{im}\Lambda.$$

Theorem 2.2. *Let χ be a KL-type k invariant. If χ is invariant under PL isotopy, then it is invariant under k-quasi-isotopy.*

Proof. For $k = 0$ there is nothing to prove, so assume $k \geq 1$. It suffices to show that any $\chi \in \tilde G_k$ vanishes on any k-quasi-embedding $f\colon mS^1 \to S^3$ with precisely one double point $f(p) = f(q)$. Let $P_0, \dots, P_k$ and $J_0, \dots, J_{k-1}$ be as in the definition of k-quasi-embedding, and let $\tilde J_0$ denote the subarc of J_0 with $\partial \tilde J_0 = \{p, q\}$. In

order to have enough room for general position we assume that P_i's are compact 3-manifolds (with boundary) and $\tilde{J}_0 \subset \operatorname{Int} J_0$ (this can be achieved by taking small regular neighborhoods).

Since the inclusion $P_0 \cup f(J_0) \hookrightarrow P_1$ is null-homotopic, there exists a generic PL homotopy $f'_t\colon f(mS^1) \to S^3$ such that the homotopy $f_t := f'_t f\colon mS^1 \to S^3$ satisfies $f_0 = f$, $f_t(J_0) \subset P_1$, $f_t = f$ outside J_0, and $f_1(\tilde{J}_0)$ is a small circle, bounding an embedded disk in the complement to $f_1(mS^1)$. Using the one-term (framing independence) relation, we see that any finite KL-type invariant vanishes on f_1. Hence any KL-type 1 vanishes on $f_0 = f$, which completes the proof for $k = 1$.

Before proceeding to the general case, we state the following generalization of the one-term (framing independence) relation.

Lemma 2.3. *If $f \in \mathcal{LM}_n$ has all its double points inside a ball B such that $f^{-1}(B)$ is an arc, then for any $\chi \in \tilde{G}_n$, $\chi(f) = 0$.*

Proof. By induction on n. $\square$

Proof of 2.2 (continued). Let us study the jump of our invariant $\chi \in \tilde{G}_k$ on the link maps f_t occurring at singular moments of the homotopy f'_t. Let $f_{t_1} \in \mathcal{LM}_2$ be one. Using the definition of k-quasi-isotopy, $k \geq 2$, we will now construct a generic PL homotopy $f'_{t_1,t}\colon f_{t_1}(mS^1) \to S^3$ such that the homotopy $f_{t_1,t} := f'_{t_1,t} f\colon mS^1 \to S^3$ satisfies $f_{t_1,0} = f_{t_1}$, $f_{t_1,t}(J_1) \subset P_2$, $f_{t_1,t} = f_{t_1}$ outside J_1, and $f_{t_1,1}$ takes J_1 into a ball B_1 in the complement to $f_{t_1,1}(mS^1 \setminus J_1)$.

Indeed, since $f_{t_1}(J_1) \subset P_1 \cup f(J_1)$ is null-homotopic in P_2, we can contract the 1-dimensional polyhedron $f_{t_1}(J_1 \setminus U)$, where U is a regular neighborhood of ∂J_1 in J_1, embedded by f_{t_1}, into a small ball $B'_1 \subset S^3 \setminus f_{t_1}(mS^1 \setminus J_1)$, by a homotopy $f_{t_1,t}|_{J_1}$ as required above. Joining the endpoints of $f_{t_1}(mS^1 \setminus \partial J_1)$ to those of $f_{t_1,t}(J_1 \setminus U)$ by two embedded arcs $f_{t_1,t}(U)$ in an arbitrary way (continuously depending on t), we obtain the required homotopy. Taking a small regular neighborhood of $B'_1 \cup f_{t_1,1}(J_1)$ relative to $f_{t_1,1}(\partial J_1)$ we obtain the required ball B_1.[1]

By Lemma 2.3, any finite KL-type invariant vanishes on $f_{t_1,1}$. The homotopy $f_{t_1,t}$ as well as the analogously constructed $f_{t_i,t}$ for each $f_{t_i} \in \mathcal{LM}_2$ does not change any invariant of KL-type 2, so for $k = 2$ we are done. The proof in the general case must be transparent now. For completeness, we state

2.4. The n^{th} inductive step. *For each critical level $f_{t_{i_1},\dots,t_{i_n}} \in \mathcal{LM}_{n+1}$ of any of the homotopies constructed in the previous step, there exists a generic PL homotopy $f'_{t_{i_1},\dots,t_{i_n},t}\colon f_{t_{i_1},\dots,t_{i_n}}(mS^1) \to S^3$ such that the homotopy $f_{t_{i_1},\dots,t_{i_n},t} := f'_{t_{i_1},\dots,t_{i_n},t} f_{t_{i_1},\dots,t_{i_n}}$ satisfies:*

(i) $f_{t_{i_1},\dots,t_{i_n},t} = f_{t_{i_1},\dots,t_{i_n}}$ for $t = 0$ and outside J_n;
(ii) $f_{t_{i_1},\dots,t_{i_n},t}(J_n) \subset P_{n+1}$ for each t;
(iii) $f^{-1}_{t_{i_1},\dots,t_{i_n},1}(B_{n-1}) = J_n$ for some PL 3-ball B_{n-1}. $\square$

Remark. In fact, the proof of Theorem 2.2 works under a weaker assumption and with a stronger conclusion. Namely, k-quasi-isotopy can be replaced with virtual k-quasi-isotopy (see definition in [MR1; §3]), whereas indistinguishability by KL-type

[1] Using the reformulation of the definition of 2-quasi-isotopy in [MR2; §1], the homotopy $f_{t_1,t}$ can be visualized as shifting the arcs $f(I'_j)$ onto the arcs $F(I_j)$ and then taking the image of J_1 into a ball along the track of the null-homotopy F.

k invariants (which can be thought of as a formal generalization of k-equivalence in the sense of Goussarov [Gu2], Habiro and Stanford) can be replaced with *geometric k-equivalence,* defined as follows. For each $n > 0$, let $\mathcal{LM}_{n;0}$ denote the subspace of $\mathcal{LM}_n$ consisting of the link maps l such that all singularities of l are contained in a ball B such that $l^{-1}(B)$ is an arc. We call link maps $l, l' \in \mathcal{LM}_n$ *geometrically k-equivalent* if they are homotopic in the space $\mathcal{LM}_n \cup \mathcal{LM}_{n+1;k}$, where $\mathcal{LM}_{i;k}$ for $k > 0$, $i > 0$ consists of those link maps with i singularities which are geometrically $(k-1)$-equivalent to a link map in $\mathcal{LM}_{i;0}$.

3. The Conway polynomial

Recall that the *Conway polynomial* of a link L is the unique polynomial $\nabla_L(z)$ satisfying $\nabla_{\text{unknot}} = 1$ and the crossing change formula

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z\nabla_{L_0}(z), \tag{1}$$

where L_+ and L_- differ by a positive crossing change, and L_0 is obtained by oriented smoothing of the self-intersection of the intermediate singular link L_s. Note that L_0 has one more (respectively less) component than L_+ and L_- if the intersection is a self-intersection of some component (resp. an intersection between distinct components).

The skein relation (1) shows that the coefficient at z^k in ∇_L is a type k invariant. The generalized one-term relation for finite type invariants specializes here to the equation $\nabla_L = 0$ for any split link L (i.e. a link whose components can be split in two nonempty parts by an embedded sphere). Using (1) it is now easy to see that the Conway polynomial of an m-component link L is necessarily of the form

$$\nabla_L(z) = z^{m-1}(c_0 + c_1 z^2 + \dots + c_n z^{2n})$$

for some n. By (1), $c_0(K) = 1$ for any knot K, which can be used to recursively evaluate c_0 on any link L. For example, it is immediate that $c_0(L)$ is the linking number of L for $m = 2$, and $c_0(L) = ab + bc + ca$ for $m = 3$, where a, b, c are the linking numbers of the 2-component sublinks (cf. [Liv]). For arbitrary m, it is easy to see that $c_0(L)$ is a symmetric polynomial in the pairwise linking numbers (and thus a KL-type 0 invariant); see [Le], [MV] for an explicit formula.

Proposition 3.1. *$c_k(L)$ is of KL-type $2k$.*

Proof. Let L be a c-colored link, and let $|c|$ denote the number of colors used, i.e. the cardinality of the image of c. Then the coefficient of ∇_L at $z^{|c|-1}$ is either $c_0(L)$ or 0 according as c is onto or not; in either case, it is a c-type 0 invariant. An induction on i using the skein relation (1) shows that the coefficient of ∇_L at $z^{|c|-1+i}$ is of c-type i. In the case where c is the identity, this is our assertion. □

In order to compute ∇_K for a knot K, we could consider a sequence of crossing changes in a plane diagram of K, turning K into the unknot (cf. [PS; Fig. 3.10]). Then $\nabla_K = \nabla_{\text{unknot}} + z\sum \varepsilon_i \nabla_{L_i}$, where L_i are the 2-component links obtained by smoothing the crossings and $\varepsilon_i = \pm 1$ are the signs of the crossing changes. We could further consider a sequence of crossing changes in the diagram of each L_i, involving only crossings of distinct components and turning L_i into a split link. This yields $\nabla_{L_i} = z\sum \varepsilon_{ij} \nabla_{K_{ij}}$, where K_{ij} are the knots obtained by smoothing the

crossings and $\varepsilon_{ij} = \pm 1$ are the signs of the crossing changes. Since the diagram of each K_{ij} has fewer crossings than that of K, we can express ∇_K, iterating this procedure, as $\sum_k z^{2k}(\sum_i \epsilon_{ki}\nabla_{\text{unknot}})$, where the signs $\epsilon_{ki} = \pm 1$ are determined by the above construction. Since $\nabla_{\text{unknot}} = 1$, plainly $\nabla_K = \sum_k z^{2k}\sum_i \epsilon_{ki}$. Note that this procedure shows that ∇_K is indeed a polynomial (rather than a power series) for any knot K; a similar argument works for links.

Now let L be a link, and suppose that L' is obtained from L by tying a knot K locally on one of the components. We can echo the above construction, expressing $\nabla_{L'}$ as $\sum_k z^{2k}\sum_i \epsilon_{ki}\nabla_L$, where the signs ϵ_{ki} are same as above. Thus[2] $\nabla_{L'} = \nabla_L\nabla_K$. It follows that the power series

$$\nabla^*_L(z) := \frac{\nabla_L(z)}{\nabla_{K_1}(z)\cdots\nabla_{K_m}(z)},$$

where $K_1, \dots, K_m$ are the components of L, is invariant under PL isotopy of L. Note that the above formula can be rewritten as $\nabla^*_L = \nabla_L - (\nabla_{K_1}\cdots\nabla_{K_m} - 1)\nabla^*_L$, meanwhile $\nabla_{K_1}\cdots\nabla_{K_m}$ is of the form $1 + b_1z^2 + \dots + b_nz^{2n}$ for some n, where $b_i(L) = \sum_{i=i_1+\dots+i_k} c_{i_1}(K_1)\cdots c_{i_m}(K_m)$. We find that ∇^*_L is of the form

$$\nabla^*_L(z) = z^{m-1}(\alpha_0 + \alpha_1z^2 + \alpha_2z^4 + \dots),$$

where $\alpha_i = c_i - (\alpha_{i-1}b_1 + \dots + \alpha_0b_i)$. Hence $\nabla^*_L \in \mathbb{Z}[[z]]$ (rather than just $\mathbb{Q}[[z^{\pm 1}]]$), and $c_k \equiv \alpha_k \pmod{\gcd(c_0, \dots, c_{k-1})}$. In particular, for $m = 2$ we see that $\alpha_0(L) = c_0(L)$ is the linking number, and $\alpha_1(L) = c_1(L) - c_0(L)(c_1(K_1) + c_1(K_2))$ is (cf. [Liv]) the generalized Sato–Levine invariant!

Under the *generalized Sato–Levine invariant* we mean the invariant that emerged in the work of Polyak–Viro (see [AMR]), Kirk–Livingston [KL], [Liv], Akhmetiev (see [AR]) and Nakanishi–Ohyama [NO]; see also [MR1].

Remark. For $m = 2$ we can also obtain $\alpha_1(L)$ from $c_1(L)$ by applying the projection Φ from Remark (i) in §2, with a certain choice of B_λ (cf. [KL; proof of Theorem 6.3]). This is not surprising, because c_1 has KL-type 2 by Proposition 3.1, hence $\Phi(c_1)$ is invariant under PL-isotopy by Remark (ii) in §2.

Remark. Clearly, the power series ∇^*_L is actually a polynomial if the components of L are unknotted or, more generally, have no non-local knots. Due to a splitting of the multi-variable Alexander polynomial (see §4), ∇^*_L splits into a product of a polynomial (which is a quotient of the original ∇_L) and a power series, both invariant under PL isotopy.

Theorem 3.2. *(i) For each $L\colon mS^1 \hookrightarrow S^3$ and any $n \in \mathbb{N}$ there exists an $\varepsilon_n > 0$ such that if $L'\colon mS^1 \hookrightarrow S^3$ is C^0 ε_n-close to L,*

$$\nabla^*_{L'}(z) \equiv \nabla^*_L(z) \mod (z^n).$$

(ii) ∇^*_L can be uniquely extended to all topological links in S^3, preserving (i).
(iii) The extended ∇^*_L is invariant under TOP isotopy of L.

[2] The conclusion is, of course, well-known, cf. [Lic; Prop. 16.2], but we need the argument in order to set up notation for use in the proof of Theorem 3.4.

Of course, the extended ∇^*_L need not be a rational power series for some wild links (which therefore will not be TOP isotopic to any PL links).

Proof. The coefficient d_k of ∇^*_L at z^k is of (monochromatic) type k since it is a polynomial in the coefficients of the Conway polynomials of L and its components, homogeneous of degree k with respect to the degrees (in z) of the corresponding terms. Since d_k is invariant under PL isotopy, by Theorem 2.2 it is invariant under k-quasi-isotopy. The assertions (i) and (ii) now follow from Theorem 1.3(a), and (iii) either from [MR1; Theorem 1.3] or, alternatively, from (ii) and compactness of the unit interval. $\square$

Proposition 3.1 implies that each $\alpha_k(L)$ is of KL-type $2k$. It is easy to check that $\alpha_1(L) = c_1(L) - c_0(L)\left(\sum_i c_1(K_i)\right)$ is, in fact, of KL-type 1 and that

$$\alpha_2(L) = c_2(L) - \alpha_1(L)\left(\sum_i c_1(K_i)\right) - c_0(L)\left(\sum_i c_2(K_i) + \sum_{i\neq j} c_1(K_i)c_1(K_j)\right)$$

is of KL-type 3.

Proposition 3.3. *$\alpha_2(L)$ is not of KL-type 2 for $m = 2$.*

Proof. Consider the case $\mathrm{lk}(L) = 0$, then $\alpha_2(L) = c_2(L) - c_1(L)(c_1(K_1) + c_1(K_2))$, which can be also written as $a_5(L) - a_3(L)(a_2(K_1) + a_2(K_2))$, where $a_i(L)$ denotes the coefficient of ∇_L at z^i. The "third derivative" of $\alpha_2(L)$, i.e. the restriction of $\bar\alpha_2(L)$ to $\mathcal{LM}^2_3$ can be found using the Leibniz rule $(\chi\psi)(L_s) = \chi(L_+)\psi(L_s) + \chi(L_s)\psi(L_-)$. In the case where the 3 singular points are all on the same component K_{sss}, it is given by

$$\bar\alpha_2(L_{sss}) = a_2(L_{000}) - a_2(L_{++0})a_0(K_{00-}) - a_2(L_{+0+})a_0(K_{0-0}) - a_2(L_{0++})a_0(K_{-00})$$

where, as usual, $+$ and $-$ stand for the overpass and the underpass, and 0 for the smoothing of the crossing s. Assuming that each L_{***} on the right hand side has three components and each K_{***} only one component, we can simplify this as $c_0(L_{000}) - c_0(L_{++0}) - c_0(L_{0+0}) - c_0(L_{00+})$.

Let $\varphi\colon S^1 \looparrowright S^2$ be a generic C^1-approximation with 3 double points of the clockwise double cover $S^1 \to S^1 \subset S^2$, and let A, B, C, D denote the 4 bounded components of $S^2 \setminus \varphi(S^1)$ such that D contains the origin. Let $K_{sss} \in \mathcal{LM}^1_3$ be the composition of φ and the inclusion $S^2 \subset S^3$, and let $K\colon S^1 \hookrightarrow S^3 \setminus K_{sss}(S^1)$ be a knot in the complement of K_{sss} linking the clockwise oriented boundaries of A, B, C, D with linking numbers a, b, c, d such that $a+b+c+2d = 0$. Finally, define $L_{sss} \in \mathcal{LM}^2_3$ to be the union of K and K_{sss}, then $\mathrm{lk}(L_{sss}) = a+b+c+2d = 0$ and we find that

$$\bar\alpha_2(L_{sss}) = (a+b+c+d)d - (a+c+d)(b+d) - (a+b+d)(c+d) - (b+c+d)(a+d).$$

This expression is nonzero e.g. for $a = 1$, $b = 2$, $c = -3$, $d = 0$. $\square$

By Proposition 3.1 and Theorem 2.2, $\alpha_n(L)$ is invariant under $2n$-quasi-isotopy. However, according to Proposition 3.3, the following strengthening of this assertion cannot be obtained by means of Theorem 2.2.

Theorem 3.4. *$\alpha_n(L)$ is invariant under n-quasi-isotopy.*

The proof makes use of the following notion. We define *colored link homotopy* to be the equivalence relation on the set of links colored with m colors, generated by intersections between components of the same color (including self-intersections) and addition of trivial component of any color, separated from the link by an embedded sphere. Thus an m-component colored link L is colored link homotopic with an $(m+k)$-component colored link L' iff L' is homotopic to $L \sqcup T_n$ through colored link maps, where T_n is the k-component unlink, split from L by an embedded sphere and colored in some way. Such a homotopy will be called a colored link homotopy between L and L'.

Proof. Let us start by considering the above procedure for computing the Conway polynomial of a knot K in more detail. One step of this procedure yields $c_k(K) = c_k(\text{unknot}) + \sum \varepsilon_i c_k(L_i)$ and $c_k(L_i) = \sum \varepsilon_{ij} c_{k-1}(K_{ij})$. Since $c_0 = 1$ for every knot, $c_k(K)$ can be computed in k steps, regardless of the number of crossings in the diagram of K. Moreover, if one is only interested in finding $c_k(K)$ for a given k (which would not allow e.g. to conclude that ∇_K is a polynomial not just a power series), the computation could be based on arbitrary generic PL homotopies rather than those suggested by the diagram of K. In particular, we allow (using that $c_0 = 0$ for every link with > 1 components and each $c_i = 0$ for a trivial link with > 1 components) self-intersections of components in the homotopies from L_i's to split links, so that K_{ij}'s may have three and, inductively, any odd number of components. For such an n step procedure, the equality $\nabla_K = \sum_{k \le n} z^{2k} \sum_i \epsilon_{ki}$, where the signs ϵ_{ki} are determined by the homotopies, holds up to degree $2n$.

Now let L be an m-component link, and suppose that L' is obtained from L by a generic PL link homotopy H. Color the components of L with distinct colors, then the $(m+1)$-component smoothed singular links L_i of the homotopy H are naturally colored with m colors. Suppose that each L_i is colored link homotopic to L; then the smoothed singular links L_{ij} of a generic PL colored link homotopy H_i between L_i and L have m or $m+2$ components colored with m colors. For a link L_* let $m(L_*)$ denote the number of components of L_* which do not (geometrically) coincide with some component of L; thus $m(L_i) = 2$ and $m(L_{ij}) = 1$ or 3, whereas $m(L) = 0$. Suppose inductively that $L_{i_1 \dots i_{k+1}}$ is one of the smoothed singular links in a generic PL colored link homotopy $H_{i_1 \dots i_k}$ between the link $L_{i_1 \dots i_k}$ and some link $L'_{i_1 \dots i_k}$ such that $m(L'_{i_1 \dots i_k}) = m(L_{i_1 \dots i_k}) - 1$. In other words, the homotopy either splits off some component (unless $L_{i_1 \dots i_k}$ has only m components) or takes some component onto a component of L (unless L is a geometric sublink of $L_{i_1 \dots i_k}$).

If there exist homotopies $H_{i_1 \dots i_l}$ as above for $l = 0, \dots, 2k$, we say that L and L' are *skein k-quasi-isotopic*. To see that α_n is invariant under skein $(n - \frac{1}{2})$-quasi-isotopy, it suffices to consider the case where there is only one singular link L_1 in the original link homotopy H. Say the double point of L_1 is on the j^{th} component; then let K and K' denote the j^{th} components of L and L'. As in the above argument for knots, we have $\nabla_{K'} = \sum_{k \le n} z^{2k} (\sum_i \epsilon_{ki} \nabla_K) + z^{2n+2} P$ and $z^{1-m} \nabla_{L'} = \sum_{k \le n} z^{2k} (\sum_i \epsilon_{ki} z^{1-m} \nabla_L) + z^{2n+2} Q$, where $P(z)$ and $Q(z)$ are some polynomials, and the signs ϵ_{ki} are determined by the homotopies $H_{i_1 \dots i_k}$. Then

$$\frac{z^{1-m}\nabla_{L'}}{\nabla_{K'}} = \frac{z^{1-m}\nabla_L R + z^{2n+2}P}{\nabla_K R + z^{2n+2}Q} = \frac{z^{1-m}\nabla_L}{\nabla_K} + z^{2n+2}S,$$

where $R(z)=\sum_{k\le n} z^{2k}\sum_i \epsilon_{ki}$ and $S(z)$ is some power series, and the assertion follows.

To complete the proof, we show that n-quasi-isotopy implies skein n-quasi-isotopy. It suffices to consider a crossing change on a component K of L, satisfying the definition of n-quasi-isotopy. Let f, $J_0,\dots,J_n$ and $P_0,\dots,P_n$ be as in the definition of n-quasi-embedding; we can assume that P_1 contains a regular neighborhood of $f(J_0)$ containing $L(J_0)$. We associate to every link L_*, such that $L\setminus K$ is a geometric sublink of L_*, but L itself is not, a collection of $m(L_*)$ positive integers $d(L_*)=(d_0,\dots,d_{m(L_*)-1})$, where d_i for $i>0$ (resp. $i=0$) is the minimal number such that the i^{th} component of L_* not in $L\setminus K$ is null-homotopic in P_{d_i} (resp. is homotopic to K with support in P_{d_i}). It is easy to see that the pair $(m(L_{i_1\dots i_{k+1}});d(L_{i_1\dots i_{k+1}}))$ is obtained from $(m(L_{i_1\dots i_k});d(L_{i_1\dots i_k}))$ by one of the following two operations:

$$(m;(d_0,\dots,d_i,\dots,d_m))\overset{\alpha_i}{\mapsto}(m+1;(d_0,\dots,d_i+1,d_i+1,\dots,d_m));$$
$$(m;(d_0,\dots,d_i,\dots,d_j,\dots,d_m))\overset{\beta_{ij}}{\mapsto}(m-1;(d_0,\dots,\max(d_i,d_j),\dots,\hat{d}_j,\dots,d_m)).$$

Conversely, if for some i the operation α_i and all operations β_{ij}'s lead to collections of integers not exceeding n, one can construct a homotopy $H_{i_1\dots i_k}$ between $L_{i_1\dots i_k}$ and some $L'_{i_1\dots i_k}$ such that $m(L'_{i_1\dots i_k})=m(L_{i_1\dots i_k})-1$.

Suppose that none of the integers $d(L_{i_1\dots i_k})$ exceeds $l+1$, and at least r of them do not exceed l. If $r\ge 1$, let d_i be one of these r, then none of the integers $\gamma(d(L_{i_1\dots i_k}))$, where γ is α_i or β_{ij}, exceeds $l+1$, and at least $r-1$ of them do not exceed l. Thus if $m(L_{i_1\dots i_k})\ge 2$ and none of the integers $d(L_{i_1\dots i_k})$ exceeds l, we can construct a homotopy $H_{i_1\dots i_k}$ as above, and for each singular link $L_{i_1\dots i_{k+1}}$ in this homotopy also a homotopy $H_{i_1\dots i_{k+1}}$ as above, so that for each singular link $L_{i_1\dots i_{k+2}}$ in this homotopy, none of the integers $d(L_{i_1\dots i_{k+2}})$ exceeds $l+1$. But $m(L_{i_1\dots i_k})\ge 2$ is indeed the case for $k=1$, hence for any odd k. $\square$

Since $c_k\equiv\alpha_k \pmod{\gcd(c_0,\dots,c_{k-1})}$, Theorem 3.4 implies the case $l=1$, and also the 2-component case of

Theorem 3.5. [MR2; Corollary 3.5] *Set* $\lambda=\lceil\frac{(l-1)(m-1)}{2}\rceil$. *The residue class of* $c_{\lambda+k}$ *modulo* gcd *of* $c_\lambda,\dots,c_{\lambda+k-1}$ *and all* $\bar\mu$*-invariants of length* $\le l$ *is invariant under* $(\lfloor\frac{\lambda}{m-1}\rfloor+k)$*-quasi-isotopy.*

(Here $\lceil x\rceil=n$ *if* $x\in[n-\frac12,n+\frac12)$, *and* $\lfloor x\rfloor=n$ *if* $x\in(n-\frac12,n+\frac12]$ *for* $n\in\mathbb{Z}$.*)*

One special case not covered by Theorem 3.4 asserts that for 3-component links the residue class of c_k modulo the greatest common divisor Δ_k of all $\bar\mu$-invariants of length $\le k+1$ is invariant under $\lfloor\frac{k}{2}\rfloor$-quasi-isotopy. Naturally, one could wonder whether an integer invariant of $\lfloor\frac{k}{2}\rfloor$-quasi-isotopy of 3-component links, congruent to $c_k \pmod{\Delta_k}$, can be found among coefficients of rational functions in ∇^* of the link and its two-component sublinks. It turns out that this is not the case[3] already for $k=1$. Indeed, we would get an integer link homotopy invariant of (monochromatic) finite type (specifically, of type 4), which is not a function of the pairwise linking numbers (since $c_1\equiv\mu(123)^2 \pmod{\Delta_1}$, cf. [Le], [MV]). But this is impossible for 3-component links by [MT].

[3] A stronger result is obtained in Proposition 4.12 below.

Alternatively, one can argue directly as follows. Consider the invariant $\gamma(L) := \alpha_1(L) - \sum \alpha_0(L_0)\alpha_1(L_1)$, where the summation is over all ordered pairs (L_0, L_1) of distinct 2-component sublinks of L. (Recall that in the 2-component case $\alpha_0(L)$ and $\alpha_1(L)$ coincide with the linking number and the generalized Sato-Levine invariant, respectively.) It can be easily verified that $\gamma(L)$ jumps by

$$\pm\mu(12)\left(\mu(1,3^+)\mu(2,3^-) + \mu(1,3^-)\mu(2,3^+)\right)$$

on any singular link $L_s = K_1 \cup K_2 \cup K_s$ with smoothing $K_1 \cup K_2 \cup K_{3^+} \cup K_{3^-}$. One can check that this jump cannot be cancelled by the jump of any polynomial expression in $c_0(L)$ and the coefficients of the Conway polynomials of the sublinks of L, homogeneous of degree 4.

Remark. An integer invariant of link homotopy of 3-component links is given by $q\Delta_1 + r$, where q is any polynomial in $\frac{\mu(ij)}{\Delta_1}$ and $r \equiv \bar\mu(123)^2 \pmod{\Delta_1}$, $0 \le r < \Delta_1$.

4. The multi-variable Alexander polynomial

The Conway polynomial is equivalent to the monochromatic case of the *Conway potential function* Ω_L of the colored link L, namely $\nabla_L(x - x^{-1}) = (x - x^{-1})\Omega_L(x)$ for monochromatic L. For a link L colored with n colors, $\Omega_L \in \mathbb{Z}[x_1^{\pm1}, \dots, x_n^{\pm1}]$ (the ring of Laurent polynomials) if L has more than one component; otherwise Ω_L belongs to the fractional ideal $(x - x^{-1})^{-1}\mathbb{Z}[x^{\pm1}]$ in the field of fractions of $\mathbb{Z}[x^{\pm1}]$.

$\Omega_L(x_1, \dots, x_n)$ is a normalized version of the sign-refined Alexander polynomial $\Delta_L(t_1, \dots, t_n)$, which is well-defined up to multiplication by monomials $t_1^{i_1} \dots t_n^{i_n}$. If K is a knot, $(x - x^{-1})\Omega_K(x) = x^\lambda \Delta_K(x^2)$, whereas for a link L with $m > 1$ components, $\Omega_L(x_1, \dots, x_n) = x_1^{\lambda_1} \dots x_n^{\lambda_n} \Delta_L(x_{c(1)}^2, \dots, x_{c(m)}^2)$, where $c\colon \{1, \dots, m\} \twoheadrightarrow \{1, \dots, n\}$ is the coloring, and the integers $\lambda, \lambda_1, \dots, \lambda_n$ are uniquely determined by the symmetry relation $\Omega_L(x_1, \dots, x_n) = (-1)^m \Omega_L(x_1^{-1}, \dots, x_n^{-1})$. We refer to Hartley [Ha], Traldi [Tr2] and Turaev [Tu] for definition of the sign-refined Alexander polynomial and discussion of the Conway potential function.

Remark (not used in the sequel). The asymmetry of the Alexander polynomial, which forces one to work with its symmetrized version Ω_L, goes back to the asymmetry of presentations of the Alexander module or, equivalently, of the Fox differential calculus. The Leibniz rule for the Fox derivative (restricted to the group elements)

$$D(fg) = D(f) + fD(g)$$

arises geometrically from considering lifts of the generators of the fundamental group of the link complement to the universal abelian cover, *starting* at a fixed lift $\tilde p$ of the basepoint. If we wish to base the whole theory on the lifts *ending* at $\tilde p$, we will have to change the Leibniz rule to

$$D(fg) = D(f)g^{-1} + D(g).$$

The "symmetric" Leibniz rule

$$D(fg) = D(f)g^{-1} + fD(g) + (f^{-1}g^{-1} - g^{-1}f^{-1})$$

does not seem to have a clear geometric meaning. (Note the analogy with the three first-order finite differences.) However, it does correspond to a "symmetric" version of the Magnus expansion, which will be behind the scenes in this section. Indeed, its abelian version gives rise, in Theorem 4.2 below, to a power series $\mho_L$, in the same way as the abelian version of the usual Magnus expansion leads to Traldi's parametrization of Ω_L, discussed in the proof of Theorem 4.8 below.

Lemma 4.1. (compare [Ki]) *(a)* $\Omega_L(x_1,\dots,x_n)=\Omega_L(-x_1^{-1},\dots,-x_n^{-1})$.

(b) If L has $m>1$ components, the total degree of every nonzero term of Ω_L is congruent to m (mod 2)*.*

(c) If L has >1 components, the x_i-degree of every nonzero term of Ω_L is congruent mod2 *to the number k_i of components of color i plus $l_i:=\sum \mathrm{lk}(K,K')$, where K runs over all components of color i and K' over those of other colors.*

Proof. It is well-known that, if all components have distinct colors, the integer λ_i from the above formula relating Ω_L to the Alexander polynomial is *not* congruent to l_i mod 2 (cf. [Tr2]). This proves (c), which implies (b). For links with >1 components (a) follows from (b) and the symmetry relation from the definition of Ω_L; for knots — from the relation of $\Omega_L(z)$ with the Conway polynomial. $\square$

Computation of Ω_L is much harder than that of ∇_L, for the skein relation (1) is no longer valid for arbitrary crossing changes. It survives, in the form

$$\Omega_{L_+}-\Omega_{L_-}=(x_i-x_i^{-1})\Omega_{L_0}, \tag{2}$$

in the case where both strands involved in the intersection are of color i. There are other formulas for potential functions of links related by local moves, which suffice for evaluation of Ω on all links; see references in [Mu].

Dynnikov [Dy] and H. Murakami [Mu] noticed that the coefficients of the power series $\Omega_L(e^{h_1/2},\dots,e^{h_n/2})$ are (monochromatic) finite type invariants of L.

Theorem 4.2. *For a link L colored with n colors there exists a unique power series $\mho_L\in\mathbb{Q}[[z_1,\dots,z_n]]$ (unless L is a knot, in which case $\mho_L\in z^{-1}\mathbb{Z}[z]$) such that*

$$\mho_L(x_1-x_1^{-1},\dots,x_n-x_n^{-1})=\Omega_L(x_1,\dots,x_n).$$

The coefficient of $\mho_L$ at a term of total degree n is of type $n+1$; moreover, $\mho_L$ satisfies the skein relation

$$\mho_{L_+}-\mho_{L_-}=z_i\mho_{L_0}$$

for intersections between components of color i. Furthermore, the total degree of every nonzero term of $\mho_L$ is congruent mod2 *to the number of components of L, and $\mho_L(4y_1,\dots,4y_n)\in\mathbb{Z}[[y_1,\dots,y_n]]$ (unless L is a knot).*

For $n=1$ we see that $z\mho_L(z)$ coincides with the (finite) polynomial $\nabla_L(z)$.

Proof. Let $x(z)$ denote the power series in z, obtained by expanding the radical in either of $x_\pm(z)=\pm\sqrt{1+z^2/4}+z/2$ by the formula $(1+t)^r=1+rt+\frac{r(r-1)}{2}t^2+\dots$. The Galois group of the quadratic equation $z=x-x^{-1}$ acts on its roots by $x\mapsto -x^{-1}$, so Lemma 4.1(a) implies that the coefficients of the power series

$$\mho_L(z_1,\dots,z_n):=\Omega_L(x(z_1),\dots,x(z_n))$$

are independent of the choice of the root $x_\pm$.

The second assertion follows by Murakami's proof [Mu] of the result mentioned above, and the third from the skein relation (2). Next, since $x^{-1}-x=-(x-x^{-1})$,

$$\begin{aligned}\mho_L(-z_1,\dots,-z_n)&=\Omega_L(x(z_1)^{-1},\dots,x(z_n)^{-1})\\&=(-1)^m\Omega_L(x(z_1),\dots,x(z_n))=(-1)^m\mho_L(z_1,\dots,z_n),\end{aligned}$$

which proves the assertion on total degrees of nonzero terms.

The final assertion may be not obvious (since $(k+1)\nmid\binom{2k-1}{k}$ for $k=1,3,7$) from

$$(1+4y)^{1/2}=1+4y\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}\binom{2k-1}{k}y^k.$$

However, this power series does have integer coefficients, since this is the case for

$$(1+4y)^{-1/2}=1+2\sum_{k=1}^{\infty}(-1)^k\binom{2k-1}{k}y^k.$$

(In fact, we see that $(k+1)\mid 2\binom{2k-1}{k}$, and $x_\pm(y)\equiv 1 \pmod 2$.) $\square$

Let us consider the case of two colors in more detail. Let c_{ij} denote the coefficient of $\mho_L(z_1,z_2)$ at $z_1^i z_2^j$. Since $\mho(z,z)=z^{-1}\nabla(z)$, c_{00} coincides with the linking number c_0 in the 2-component case and is zero otherwise. The skein relation (2) implies that for a 3-component link $K_1\cup K_1'\cup K_2$, colored as indicated by subscript, c_{10} coincides with $\operatorname{lk}(K_1,K_1')(\operatorname{lk}(K_1,K_2)+\operatorname{lk}(K_1',K_2))$ up to a type 0 invariant of the colored link. The latter is identically zero by the connected sum[4] formula

$$\Omega_{L\#_i L'}=(x_i-x_i^{-1})\Omega_L\Omega_{L'},\tag{3}$$

which follows from the definition of Ω_L in [Ha]. Since $c_{01}=c_0-c_{10}$ or by an analogous argument, $c_{01}=\operatorname{lk}(K_1,K_2)\operatorname{lk}(K_1',K_2)$. It follows that for 2-component links c_{11} coincides with the generalized Sato–Levine invariant α_1 up to a KL-type 0 invariant. Since α_1-c_{11} is of type 3, it has to be a degree 3 polynomial in the linking number, which turns out to be $\frac{1}{12}(\operatorname{lk}^3-\operatorname{lk})$. (Since $\Omega_{\text{unlink}}=0$ and $\Omega_{\text{Hopf link}}=1$, the second Conway identity [Ha], [Ki] can be used to evaluate Ω_L on a series of links $\mathcal{H}_n$ with $\operatorname{lk}\mathcal{H}_n=n$.) The half-integer c_{11} is thus the *unoriented generalized Sato–Levine invariant*, which is implicit in the second paragraph of [KL]. Specifically, $2c_{11}/\operatorname{lk}^2$ is (unless $\operatorname{lk}=0$) the Casson–Walker invariant of the $\mathbb{Q}$-homology sphere obtained by 0-surgery on the components of the link; in fact, $c_{11}(L)=\frac{\alpha_1(L)+\alpha_1(L')}{2}$, where L' denotes the link obtained from L by reversing the orientation of one of the components [KL; proof of Theorem F].

The preceding paragraph implies the first part of the following

Theorem 4.3. *Consider the power series*

$$\mho_L^*(z_1,\dots,z_n):=\frac{\mho_L(z_1,\dots,z_n)}{\nabla_{K_1}(z_{c(1)})\cdots\nabla_{K_m}(z_{c(m)})},$$

where $K_1,\dots,K_m$ denote the components of L. Let α_{ij} denote the coefficient of $\mho_L^*(z_1,z_2)$ at $z_1^i z_2^j$. For 2-component links

(i) $\alpha_{00}=\operatorname{lk}$, and α_{11} is the unoriented generalized Sato–Levine invariant, which assumes all half-integer values;

[4] The multi-valued operation of Hashizume connected sum $L\#_i L'$ is defined as follows. Let $L'(mS^1)$ be split from $L(mS^1)$ by an embedded S^2, and $b\colon I\times I\hookrightarrow S^3$ be a band meeting the embedded S^2 in an arc, and $L(mS^1)$ (resp. $L'(mS^1)$) in an arc of color i, identified with $b(I\times\{0\})$ (resp. $b(I\times\{1\})$). Then $L\#_i L'(mS^1)=((L(mS^1)\cup L'(mS^1))\setminus b(I\times\partial I))\cup b(\partial I\times I)$.

(ii) $\alpha_{1,2k-1}$ and $\alpha_{2k-1,1}$ coincide, when multiplied by $(-1)^{k+1}$, with Cochran's [Co] *derived invariants $\beta^k_\pm$, whenever the latter are defined (i.e.* $\mathrm{lk}=0$*);*

(iii) α_{ij} is of type $i+j$; when $i+j$ is odd, $\alpha_{ij}=0$.

The last part is an immediate corollary of Theorem 4.2.

Proof of (ii). It follows from [Co; Theorem 7.1] that $\sum_{i=1}^\infty \beta^i_L(-z^2)^i=\eta_L(y^2)$, where η_L is Kojima's η-function and $z=y-y^{-1}$. On the other hand, it follows from [Jin; Theorem 4] that

$$\eta_L(y^2)=\pm\frac{(1-y^2)(1-y^{-2})\Omega'_L(1,y)}{\nabla_{K_2}(y-y^{-1})},$$

where Ω'_L is given by $\Omega_L(x_1,x_2)=(x_1-x_1^{-1})(x_2-x_2^{-1})\Omega'_L(x_1,x_2)$ for 2-component links with $\mathrm{lk}=0$. It is shown in [Jin; Lemma 8] (see also proof of Theorem 4.8(ii) below) that under this assumption Ω'_L is a Laurent polynomial (rather than just a rational function). Define $\mho'_L$ by $\mho_L(z_1,z_2)=z_1z_2\mho'_L(z_1,z_2)$ for 2-component links with $\mathrm{lk}=0$, then

$$\sum_{i=1}^\infty(-1)^i\beta^i_Lz^{2i}=\mp\frac{z^2\mho'_L(0,z)}{\nabla_{K_2}(z)},$$

and consequently

$$\sum_{i=0}^\infty(-1)^i\beta^{i+1}_Lz^{2i}=\pm\frac{\mho'_L(0,z)}{\nabla_{K_1}(0)\nabla_{K_2}(z)}.$$

According to (i), the sign must be positive. $\square$

It turns out that each coefficient of the power series $\mho$ can be canonically split into a $\mathbb{Q}$-linear combination of the coefficients of certain 2^{n-1} polynomials.

Theorem 4.4. *The Conway potential function of any colored link L with >1 components can be uniquely written in the form*

$$\Omega_L(x_1,\dots,x_n)=\sum_{1\le i_1<\dots<i_{2k}\le n}\left\{\frac{x_{i_1}}{x_{i_2}}\cdots\frac{x_{i_{2k-1}}}{x_{i_{2k}}}\right\}P_{i_1\dots i_{2k}}(\{x_1\},\dots,\{x_n\})\qquad(4)$$

for some $P_{i_1\dots i_{2k}}\in\frac12\mathbb{Z}[z_1,\dots,z_n]$, where $\{f(x_1,\dots,x_n)\}$ denotes $f(x_1,\dots,x_n)+f(-x_1^{-1},\dots,-x_n^{-1})$ for any function $f(x_1,\dots,x_n)$. Moreover, if $n=2l$, the coefficients of $P_{12\dots n}$ are integer.

Here is the case $n=2$ in more detail:

$$\Omega(x,y)=2P(x-x^{-1},y-y^{-1})+(xy^{-1}+x^{-1}y)P_{12}(x-x^{-1},y-y^{-1}).$$

This case was essentially known in 1986; indeed it is equivalent (see formula $(*)$ below) to Kidwell's decomposition [Ki; Theorem 2]:

$$\Omega(x,y)=K_1(x-x^{-1},y-y^{-1})+(xy+x^{-1}y^{-1})K_2(x-x^{-1},y-y^{-1}).$$

However, the proof of Theorem 4.8(iii) below (on $\bar\mu$-invariants) breaks down for K_1 and K_2 in place of $2P$ and P_{12}. Moreover, the assertion on integrality of $2P_{i_1\dots i_{2k}}$

in Theorem 4.4 will not hold already for $n = 3$ (resp. $n = 5$) if $\{\frac{x_{i_1}}{x_{i_2}} \cdots \frac{x_{i_{2k-1}}}{x_{i_{2k}}}\}$ is replaced with $\{x_{i_1} \cdots x_{i_{2k}}\}$ (resp. with $\{\frac{x_{i_1} \cdots x_{i_k}}{x_{i_{k+1}} \cdots x_{i_{2k}}}\}$) in (4), at least for some Laurent polynomial Ω satisfying the conclusion of Lemma 4.1(a). (Lemma 4.1(a) is the only property of Ω_L used in the proof of Theorem 4.4.)

Proof. By Lemma 4.1(a), Ω_L includes together with every term $Ax_1^{p_1} \cdots x_n^{p_n}$ the term $(-1)^{p_1+\cdots+p_n} Ax_1^{-p_1} \cdots x_n^{-p_n}$, and so can be written as a $\mathbb{Z}$-linear combination of the L-polynomials $\{x_1^{p_1} \cdots x_n^{p_n}\}$. The formula

$$\{x_i M\} - \{x_i^{-1} M\} = \{x_i\}\{M\}, \tag{$*$}$$

where M is any monomial in $x_1^{\pm 1}, \dots, x_n^{\pm 1}$, can be verified directly (separately for M of odd and even total degrees), and allows to express each $\{x_1^{p_1} \dots x_n^{p_n}\}$ in the form

$$\sum_{1 \le i_1 < \cdots < i_k \le n} \{x_{i_1} x_{i_2}^{-1} x_{i_3} x_{i_4}^{-1} \cdots (x_{i_{k-1}} x_{i_k}^{-1})^{(-1)^k}\} P'_{i_1 \dots i_k}(\{x_1\}, \dots, \{x_n\})$$

for some $P'_{i_1 \dots i_k} \in \mathbb{Z}[z_1, \dots, z_n]$, $k > 0$, and some $P' \in \frac{1}{2}\mathbb{Z}[z_1, \dots, z_n]$. The summands corresponding to $k = 1$ can be included in P', and one can get rid of the summands corresponding to odd $k \ge 3$ by repeated use of the formula

$$2\{x_i x_j^{-1} M\} = \{x_i\}\{x_j^{-1} M\} + \{x_j^{-1}\}\{x_i M\} + \{x_i x_j\}\{M\}, \tag{$**$}$$

combined with $(*)$. Note that $(**)$ follows immediately from $(*)$ and its analogue

$$\{x_i x_j M\} + \{x_i^{-1} x_j^{-1} M\} = \{x_i x_j\}\{M\}.$$

But it is not clear from this approach that the resulting polynomials have half-integer coefficients. To see this, represent $2\{x_{i_1} x_{i_2}^{-1} x_{i_3} \cdots x_{i_{2k}}^{-1} x_{i_{2k+1}}\}$ as

$$\begin{aligned} &\{x_{i_1} x_{i_2}^{-1} x_{i_3} \cdots x_{i_{2k}}^{-1} x_{i_{2k+1}}\} - \{x_{i_1}^{-1} x_{i_2} x_{i_3}^{-1} \cdots x_{i_{2k}} x_{i_{2k+1}}^{-1}\} \\ &+ (1-1) \sum_{j=1}^{2k} \{x_{i_1}^{-1} x_{i_2} x_{i_3}^{-1} \cdots (x_{i_{j-2}} x_{i_{j-1}}^{-1} x_{i_j} x_{i_{j+1}} x_{i_{j+2}}^{-1} x_{i_{j+3}})^{(-1)^j} \cdots x_{i_{2k}}^{-1} x_{i_{2k+1}}\}. \end{aligned}$$

Then formula $(*)$ yields: $\quad 2\{x_{i_1} x_{i_2}^{-1} x_{i_3} \cdots x_{i_{2k}}^{-1} x_{i_{2k+1}}\}$

$$= \sum_{j=1}^{2k+1} (-1)^{j+1} \{x_{i_j}\}\{x_{i_1}^{-1} x_{i_2} x_{i_3}^{-1} \cdots (x_{i_{j-2}} x_{i_{j-1}}^{-1} x_{i_{j+1}} x_{i_{j+2}}^{-1} x_{i_{j+3}})^{(-1)^j} \cdots x_{i_{2k}}^{-1} x_{i_{2k+1}}\}.$$

Thus each $P'_{i_1 \dots i_{2k+1}}$ can be included in the polynomials $\frac{1}{2} P'_{j_1 \dots j_{2k}}$.

It remains to verify uniqueness of (4). Suppose, by way of contradiction, that a nontrivial expression $Q(x_1, \dots, x_n)$ in the form of the right hand side of (4) is identically zero. Then so is $Q(x_1, \dots, x_{n-1}, x_n) - Q(x_1, \dots, x_{n-1}, -x_n^{-1})$, which can be rewritten as

$$[x_n] \sum_{1 \le i_1 < \cdots < i_{2k-1} \le n-1} [x_{i_1} x_{i_2}^{-1} x_{i_3} \cdots x_{i_{2k-2}}^{-1} x_{i_{2k-1}}] P_{i_1 \dots i_{2k-1}, n}(\{x_1\}, \dots, \{x_n\}) = 0,$$

where $[f(x_1,\dots,x_n)]$ denotes $f(x_1,\dots,x_n)-f(-x_1^{-1},\dots,-x_n^{-1})$. Denote the left hand side by $[x_n]R(x_1,\dots,x_n)$, then $R(x_1,\dots,x_n)$ is identically zero. Hence so is $R(x_1,\dots,x_{n-2},x_{n-1},x_n)-R(x_1,\dots,x_{n-2},-x_{n-1}^{-1},x_n)$, which can be rewritten as

$$[x_{n-1}]\sum_{1\le i_1<\dots<i_{2k}\le n-2}\{x_{i_1}x_{i_2}^{-1}\cdots x_{i_{2k-1}}x_{i_{2k}}^{-1}\}P_{i_1\dots i_{2k},n-1,n}(\{x_1\},\dots,\{x_n\})=0.$$

Repeating this two-step procedure $\lfloor\frac{n}{2}\rfloor$ times, we will end up with

$$[x_1]\{1\}P_{1\dots n}(\{x_1\},\dots,\{x_n\})=0\quad\text{ or }\quad[x_2]\{1\}P_{2\dots n}(\{x_1\},\dots,\{x_n\})=0$$

according as n is even or odd. Consider, for example, the case of odd n. By symmetry, $P_{1\dots\hat{\imath}\dots n}=0$ for each i. Returning to the previous stage

$$[x_4]\sum_{1\le i_1<\dots<i_{2k}\le 3}\{x_{i_1}x_{i_2}^{-1}\cdots x_{i_{2k-1}}x_{i_{2k}}^{-1}\}P_{i_1\dots i_{2k},4\dots n}(\{x_1\},\dots,\{x_n\})=0,$$

we can now substitute zeroes for $P_{23,4\dots n},P_{13,4\dots n},P_{12,4\dots n}$, and so we get $P_{4\dots n}=0$. Continuing to the earlier stages, we will similarly verify that each $P_{i_1\dots i_{2k}}=0$. $\square$

Since every term of, say, $x-x^{-1}$ has an odd degree in x, Lemma 4.1 implies

Lemma 4.5. *For every nonzero term of $P_{i_1\dots i_k}$*
(i) the total degree is congruent mod2 *to the number of components;*
(ii) the z_i-degree is congruent to $k_i+l_i \pmod 2$ iff $i\notin\{i_1,\dots,i_k\}$.

Let us turn again to the case of two-component links colored with two colors. Let d'_{ij} (resp. d''_{ij}) denote the coefficient of $P(z_1,z_2)$ (resp. $P_{12}(z_1,z_2)$) at $z_1^iz_2^j$. Substituting $x(z_i)$ for x_i as in the proof of Theorem 4.2, we get

$$x_1x_2^{-1}+x_1^{-1}x_2=2-\frac{z_1z_2}{2}+\frac{z_1^2}{4}+\frac{z_2^2}{4}+\text{ terms of total degree}\ge 4.$$

Thus $c_{00}=2(d'_{00}+d''_{00})$ and

$$c_{11}=2(d'_{11}+d''_{11})-\frac{d''_{00}}{2},\quad c_{20}=2(d'_{20}+d''_{20})+\frac{d''_{00}}{4},\quad c_{02}=2(d'_{02}+d''_{02})+\frac{d''_{00}}{4}.$$

By Lemma 4.5(ii), $d'_{2i,2j}=d''_{2i+1,2j+1}=0$ if the linking number is even; otherwise $d'_{2i+1,2j+1}=d''_{2i,2j}=0$. On the other hand, $c_{00}=\operatorname{lk}$ and $c_{11}=\alpha_1-(\operatorname{lk}^3-\operatorname{lk})/12$ by Theorem 4.3(i). Thus $2d''_{00}$ is lk or 0, and $2d''_{11}$ is 0 or the integer $\alpha_1-(\operatorname{lk}^3-\operatorname{lk})/12$, according as lk is even or odd. Similarly, $2d'_{00}$ is 0 or lk, and $2d'_{11}$ is the integer $\alpha_1-(\operatorname{lk}^3-4\operatorname{lk})/12$ or 0, according as lk is even or odd. On the other hand, it follows from Proposition 2.1 that the KL-type 0 invariant $2d'_{00}=((-1)^{\operatorname{lk}}+1)\operatorname{lk}/2$ is not of finite type. Since the linking number c_{00} and the half-integer c_{11} are of finite type, the integer $d_{11}:=2(d'_{11}+d''_{11})$ is also not of finite type.

Remark. The decomposition (4) in the case $n=2$ can be modified, using the identity

$$xy^{-1}+x^{-1}y=\frac{(x-y)^2}{xy}+2,$$

to the form

$$\Omega_L(x,y) = 2(P(\{x\},\{y\}) + P_{12}(\{x\},\{y\})) + \frac{(x-y)^2}{xy} P_{12}(\{x\},\{y\}).$$

Lemma 4.5(ii) implies that each coefficient of $P+P_{12}$ coincides with the corresponding coefficient of either P or P_{12}, depending on the parity of $\mathrm{lk}(L)$. Since $\mathrm{lk}(L)$ is the constant term of $2(P+P_{12})$, the remainder $(xy)^{-1}(x-y)^2P_{12}$ is redundant, in the sense that it contains no additional information with respect to $2(P+P_{12})$.

More generally, we have

Theorem 4.6. *Let $P_{i_1,\dots,i_{2k}}$ be the polynomials of the colored link L, defined in (4). The polynomial $\triangledown_L$, defined by*

$$\triangledown_L(z_1,\dots,z_n) := 2 \sum_{1\le i_1<\dots<i_{2k}\le n} P_{i_1\dots i_{2k}}(z_1,\dots,z_n) \in \mathbb{Z}[z_1,\dots,z_n]$$

if L has at least two components, and by $\triangledown_L(z) = z^{-1}\nabla_L(z) \in z^{-1}\mathbb{Z}[z]$ if $n=1$, determines, and is determined by, the Conway potential function $\Omega_L(x_1,\dots,x_n)$. The coefficient of $\triangledown_L$ at a term of total degree d is of colored type $d+1$, and vanishes unless d is congruent $\bmod 2$ *to the number of components of L.*

Proof. By Theorem 4.4, Ω_L determines each $P_{i_1\dots i_{2k}}$, hence $\triangledown_L$. Conversely, Lemma 4.5(ii) implies that the coefficients of each $P_{i_1\dots i_{2k}}$, hence of Ω_L, are determined by $\triangledown_L$. The last two assertions follow from Lemma 4.5(i) and the skein relation (2), which can be rewritten in the form

$$\triangledown_{L_+} - \triangledown_{L_-} = z_i \triangledown_{L_0}$$

for intersections between components of color i. $\square$

Let $K_1,\dots,K_m$ denote the components of a colored link L, where $m>1$. It follows from Theorem 4.4 that the power series

$$\Omega^*_L(x_1,\dots,x_n) := \frac{\Omega_L(x_1,\dots,x_n)}{\nabla_{K_1}(x_{c(1)}-x^{-1}_{c(1)})\cdots\nabla_{K_m}(x_{c(m)}-x^{-1}_{c(m)})} \in \mathbb{Z}[[x_1^{\pm1},\dots,x_n^{\pm1}]]$$

can be expressed in the form

$$\Omega^*_L(x_1,\dots,x_n) = \sum_{1\le i_1<\dots<i_{2k}\le n} \left\{\frac{x_{i_1}}{x_{i_2}}\cdots\frac{x_{i_{2k-1}}}{x_{i_{2k}}}\right\} P^*_{i_1\dots i_{2k}}(\{x_1\},\dots,\{x_n\}),$$

where each $P^*_{i_1\dots i_{2k}} \in \mathbb{Z}[[z_1,\dots,z_n]]$, and the proof of Theorem 4.4 shows that such an expression is unique. The doubled sum of all these polynomials is, of course, nothing but

$$\triangledown^*_L(z_1,\dots,z_n) := \frac{\triangledown_L(z_1,\dots z_n)}{\nabla_{K_1}(z_{c(1)})\cdots\nabla_{K_m}(z_{c(m)})},$$

which contains the same information as Ω^*_L by the proof of Theorem 4.6.

Corollary 4.7. *(i) For each $L\colon mS^1 \hookrightarrow S^3$ and any $k \in \mathbb{N}$ there exists an $\varepsilon_k > 0$ such that if $L'\colon mS^1 \hookrightarrow S^3$ is C^0 ε_k-close to L,*

$$\triangledown^*_{L'} = \triangledown^*_L + \text{terms of total degree} \geq k,$$
$$\mho^*_{L'} = \mho^*_L + \text{terms of total degree} \geq k.$$

(ii) $\triangledown^*_L$ and $\mho^*_L$ can be uniquely extended to all TOP links in S^3, preserving (i).

(iii) The extended $\triangledown^*_L$ and $\mho^*_L$ are invariant under TOP isotopy of L.

Proof. As in the above argument for ∇^*_L, it is easy to see that $\triangledown^*_L$, as well as $\mho^*_L$, are invariant under PL isotopy (cf. [Tr2; Theorem 1]). By Theorems 4.2 and 4.6, the coefficients of $\triangledown^*_L$ and $\mho^*_L$ at terms of total degree k are of colored type $k+1$, hence by Theorem 2.2 they are invariant under $(k+1)$-quasi-isotopy. The rest of the proof is as in Theorem 3.2. □

Remark. If the components of L are unknotted, or more generally have no non-local knots, $\triangledown^*_L$ is a (finite) polynomial. Actually, $\triangledown^*_L$ splits into a product of m one-variable power series and a polynomial (namely, the product of the irreducible factors of $\triangledown_L$, involving more than one variable), so that each of them is individually invariant under PL isotopy; compare [Tr2], [Ro]. There is, of course, such a splitting for every $P^*_{i_1\dots i_{2n}}$, producing a plethora of PL isotopy invariants.

Lack of an analogue of Lemma 4.5(ii) for the power series $\mho_L$ hinders establishing a simple relation between the coefficients of $\mho_L$ and those of Traldi's expansion of Ω_L [Tr2], which prevents one from expressing the finite type rationals α_{ij} (discussed in Theorem 4.3) as liftings of $\bar\mu$-invariants. However, each α_{ij} can be split into a linear combination (whose coefficients depend on the linking number) of the integer coefficients of the power series $\triangledown^*_L$, which do admit such an expression.

Theorem 4.8. *Let δ_{ij} denote the coefficient of $\triangledown^*_L$ at $z_1^i z_2^j$. For 2-component links*

(i) $\delta_{00} = \mathrm{lk}$, and δ_{11} is an integer lifting of the Sato–Levine invariant $\bar\mu(1122)$, but not a finite type invariant;

(ii) $\delta_{1,2k-1}$ and $\delta_{2k-1,1}$ are Cochran's [Co] derived invariants $(-1)^{k+1}\beta^k_\pm$, whenever the latter are defined (i.e. $\mathrm{lk} = 0$);

(iii) δ_{ij} is an integer lifting of Milnor's [Mi] invariant $(-1)^j\bar\mu(\underbrace{1\dots1}_{i+1}\underbrace{2\dots2}_{j+1})$, provided that $i+j$ is even;

(iv) δ_{ij} is of KL-type $i+j$;

(v) when $i+j$ is odd, $\delta_{ij} = 0$;

(vi) when $i+1$, $j+1$ and lk are all even or all odd, δ_{ij} is even;

(vii) for a given L, there are only finitely many pairs (i,j) such that $\delta_{ij} \not\equiv 0$ modulo the greatest common divisor Δ_{ij} of all δ_{kl}'s with $k \leq i$, $l \leq j$ and $k+l < i+j$; the congruence can be replaced by equality if the components of L are unknotted.

The last four parts are immediate consequences of Theorem 4.6; Lemma 4.5(i); Lemma 4.5(ii) and integrality of the coefficients of P^*_{12}; the definition of $\triangledown^*_L$ as a rational power series. The first part follows from the discussion after Lemma 4.5.

Remark. The geometry of the relationship between $\bar\mu$-invariants and the multivariable Alexander polynomial is now better understood [KLW]. If L is the link closure of a string link $\ell\colon \{1,\dots,m\}\times I \hookrightarrow \mathbb{R}^2\times I$,

$$\Delta_L = \Delta_\ell \Gamma_\ell,$$

where Δ_L denotes the usual Alexander polynomial, Δ_ℓ the Alexander polynomial of the string link [LeD], i.e. the Reidemeister torsion of the based chain complex of the pair $(X, X\cap\mathbb{R}^2\times\{0\})$, where $X=\mathbb{R}^2\times I\setminus\operatorname{im}\ell$, and Γ_ℓ a certain rational power series, determined by the μ-invariants of ℓ.

Proof of (iii). Since we are only interested in the residue class of every δ_{ij} mod Δ_{ij}, we may consider the polynomial $\nabla_L = 2(P+P_{12})$ in place of the power series $\nabla_L^* = 2(P^*+P_{12}^*)$. By Theorem 4.4 and Lemma 4.5(ii), Ω_L is uniquely expressible in the form

$$(\{x_1\}\{x_2\})^\lambda\Omega_L(x_1,x_2) = Q_1(\{x_1\}^2,\{x_2\}^2) + \{x_1x_2^{-1}\}\{x_1\}\{x_2\}Q_2(\{x_1\}^2,\{x_2\}^2)$$

for some $Q_1, Q_2\in\mathbb{Z}[z_1,z_2]$, where $\lambda=0$ or 1 according as $\operatorname{lk}(L)$ is odd or even. Set $y_i = x_i^2-1$, then $x_i^{-2} = 1-y_i+y_i^2-y_i^3+\dots$. The identities

$$\{x_i\}^2 = x_i^{-2}(x_i^2-1)^2 = (1-y_i+y_i^2-\dots)y_i^2;$$
$$\{x_1x_2^{-1}\}\{x_1\}\{x_2\} = (x_1^{-2}+x_2^{-2})(x_1^2-1)(x_2^2-1) = (2-y_1-y_2+y_1^2+y_2^2-\dots)y_1y_2$$

allow to express $(x_1x_2)^{-\lambda}\Omega_L(x_1,x_2)$ as a power series $T_L(y_1,y_2)$ with integer coefficients.

Let us study this substitution more carefully. Let d'_{ij}, d''_{ij} denote the coefficients at $z_1^iz_2^j$ in P and P_{12}, and let us write $(k,l)\le(i,j)$ if $k\le i$ and $l\le j$. Then the coefficient e''_{ij} at $y_1^iy_2^j$ in the power series R_2, defined by the equality $(y_1y_2)^{\lambda-1}R_2(y_1,y_2) = (z_1z_2)^{\lambda-1}P_{12}(z_1,z_2)$, is given by

$$e''_{ij} = \sum_{(k,l)\le(i,j)}(-1)^{(i-k)+(j-l)}d''_{kl} = (-1)^{i+j}\sum_{(k,l)\le(i,j)}d''_{kl}$$

(the latter equality uses that $d''_{kl}=0$ if $k+l$ is odd), and similarly for the coefficients e'_{ij} of the power series R_1, defined by $(y_1y_2)^\lambda R_1(y_1,y_2) = (z_1z_2)^\lambda P(z_1,z_2)$. Now the coefficients e_{ij} of $T_L(y_1,y_2)$ are given by

$$\begin{aligned} e_{ij} &= 2e'_{ij}+2e''_{ij}+\sum_{k<i}(-1)^{i-k}e''_{kj}+\sum_{l<j}(-1)^{j-l}e''_{il} \\ &\qquad\qquad = 2e'_{ij}+2e''_{ij}+(-1)^{i+j}\sum_{(k,l)<(i,j)}((i-k)+(j-l))d''_{kl}. \end{aligned}$$

The key observation here is that all coefficients on the right hand side are even if $i+j$ is even.

Let us consider the case $\lambda=1$. Then by Lemma 4.5(ii), $d''_{ij}=0$ unless both i and j are even, and $d'_{ij}=0$ unless both i and j are odd. Hence $e''_{ij}\equiv 0$ modulo $\gcd\{e''_{kl}\mid(k,l)<(i,j)\}$, unless both i and j are even, and similarly for e'_{ij}. Now it follows by induction that $e_{ij}\equiv 2e''_{ij}\equiv 2d''_{ij}$ or $e_{ij}\equiv 2e'_{ij}\equiv 2d'_{ij}$ modulo $E_{ij} := \gcd\{e_{kl}\mid(k,l)<(i,j)\}$ according as i and j are both even or both odd. Thus $e_{ij}\equiv 2(d'_{ij}+d''_{ij}) \pmod{E_{ij}}$ if $i+j$ is even. Clearly, the latter assertion holds in the case $\lambda=0$ as well, which can be proved by the same argument.

Finally, since $-\lambda\equiv\operatorname{lk}-1 \pmod 2$, and $x_i^{\pm2}$ are expressible as power series in y_i with integer coefficients and constant term 1, the coefficients $\tilde e_{ij}$ of Traldi's power

series $\tilde{T}_L(y_1,y_2)=(x_1x_2)^{\operatorname{lk}-1}\Omega_L$ are related to e_{ij} by congruence modE_{ij}. This completes the proof, since by [Tr2; Corollary 8.4], each $\tilde{e}_{ij}$ is an integer lifting of $(-1)^{j+1}\bar{\mu}(\underbrace{1\dots1}_{i+1}\underbrace{2\dots2}_{j+1})$. $\square$

Remark. The above argument yields a new proof of another result due to Traldi: $2\bar{\mu}(\underbrace{1\dots1}_{i+1}\underbrace{2\dots2}_{j+1})=0$ when $i+j$ is odd [Tr1; Corollary 6.6].

Proof of (ii). We recall that $\bar{\mu}(1\dots12)$ and $\bar{\mu}(12\dots2)$ identically vanish, with the exception of $\bar{\mu}(12)=\operatorname{lk}$ [Mi]. Hence by (iii), $\delta_{0,2k}\equiv\delta_{2k,0}\equiv 0 \pmod{\operatorname{lk}}$ for each k. So if $\operatorname{lk}=0$, every nonzero term of either P^* or P^*_{12} involves both z_1 and z_2. (Alternatively, this follows from Jin's lemma mentioned in the proof of Theorem 4.3(ii).) By Lemma 4.5(ii), every nonzero term of P^*_{12} has to further include each of them once again, i.e. P^*_{12} is divisible by $z_1^2z_2^2$. Hence, firstly, $\delta_{2k-1,1}$ coincides with the coefficient of $2P^*(z_1,z_2)$ at $z_1^{2k-1}z_2$, and, secondly, this coefficient is not affected by adding $(x(z_1)x(z_2)^{-1}+x(z_2)x(z_1)^{-1})P^*_{12}(z_1,z_2)$ to $2P^*(z_1,z_2)$, where $x(z)$ is as in the proof of Theorem 4.2. $\square$

Theorem 4.9. *The coefficient of $\bigtriangledown^*_L$ at $z_1^{i_1}\dots z_n^{i_n}$ is invariant under k-quasi-isotopy if* $\max(i_1,\dots,i_n)\le 2l+1$.

Proof. This is analogous to Theorem 3.4. One only needs to show that the coefficient in question is invariant under skein l-quasi-isotopy, which is done by the same argument, replacing every occurrence of the polynomial $z^{1-m}\nabla_L$ with $\bigtriangledown_L$. $\square$

Corollary 4.10. *(a)* [MR2; Corollary 3.6] *Cochran's invariants $\beta^k_\pm$ are invariant under k-quasi-isotopy;*

(b) Milnor's invariants $\bar{\mu}(1\dots12\dots2)$ of even length are invariant under k-quasi-isotopy, if each index occurs at most $2k+2$ times.

Part (b) covers (and largely improves) the corresponding case of

Theorem 4.11. [MR; Corollary 3.4] *All $\bar{\mu}$-invariants of length $\le 2k+3$ are invariant under k-quasi-isotopy.*

We conclude by a further examination of the coefficients of the Conway polynomial. In view of the discussion following Theorem 3.5, one could wonder whether an integer lifting of the residue class of c_n mod Δ_n, invariant under $\lfloor\frac{n}{2}\rfloor$-quasi-isotopy, can be extracted from the Conway potential function Ω_L of the 3-component link L. In order to exclude the possibility of the trivial extension (by zeroes) one has to specify what it means to be extracted, and so we assume that our invariant is a polynomial expression in the coefficients of the polynomial $\bigtriangledown_L$, homogeneous of degree $2n+1$ with respect to the total degrees of the corresponding terms. Then the following shows that this is impossible.

Proposition 4.12. *No integer link homotopy invariant γ' of 3-component links, such that $\gamma'\equiv c_1 \pmod{\gcd\mu(ij)}$, is a polynomial in the coefficients of $\bigtriangledown_L$, homogeneous of degree 3 with respect to the total degrees of the corresponding terms.*

Proof. Consider the Hopf link $\mathcal{H}=K_1\cup K_2$, and let K_i' be a meridian of the boundary of a small regular neighborhood of K_i. Push 'fingers' from K_1' and K_2' towards the basepoint, so as to realize $L_0=\mathcal{H}\cup K_1'\cup K_2'$ as the smoothing of a singular 3-component link, which is the only singular link in a generic link homotopy

between $L_+ = \mathcal{H} \cup K_+$ and $L_- = \mathcal{H} \cup K_-$. Then $\Omega_{L_+} - \Omega_{L_-} = \pm\{z\}\Omega_{L_0}$ by the skein relation (2), where both K_1' and K_2' have color z, whereas K_1 has color x and K_2 has color y. By the connected sum formula (3), we have $\Omega_{L_0} = \pm\{x\}\Omega_{\mathcal{H}\cup K_1'}$ and $\Omega_{\mathcal{H}\cup K_1'} = \pm\{y\}\Omega_{\mathcal{H}}$. Since $\Omega_{\mathcal{H}} = 1$, the jump of Ω on the homotopy between L_+ and L_- is $\pm\{x\}\{y\}\{z\}$. On the other hand, the crossing change formula for γ from the end of §3 shows that γ jumps by ± 1 in this situation. Since γ' is assumed to be a link homotopy invariant, so does $\gamma - \gamma'$. Now γ' is assumed to be a function of the coefficients of $\bigtriangledown_L$, and certainly γ is such a function. (Indeed, if L' is obtained from L by deleting a component,

$$\Omega_L(1, x_1, \dots, x_n) = (x_1^{l_1} \dots x_n^{l_n} - x_1^{-l_1} \dots x_n^{-l_n})\Omega_{L'}(x_1, \dots, x_n),$$

where $l_i = \sum \mathrm{lk}(K_0, K)$ with K running over all components of L' of color i [Ha].) Thus $\gamma - \gamma'$ depends nontrivially on the coefficient d_{111} at $\{x\}\{y\}\{z\}$. By homogeneity, $\gamma - \gamma'$ has to be linear in d_{111}. Therefore, $\gamma - \gamma'$ assumes nonzero value on the Borromean rings $\mathcal{B}$, whose Conway function happens to be $\{x\}\{y\}\{z\}$. Since the pairwise linking numbers of $\mathcal{B}$ are zero, $\alpha_1(\mathcal{B}) = \gamma(\mathcal{B}) \neq \gamma'(\mathcal{B})$, which contradicts our assumption $\gamma' \equiv \alpha_1 \pmod{\gcd \mu(ij)}$. □

5. The HOMFLY and Kauffman polynomials

We recall that the *HOMFLY(PT) polynomial* and the Dubrovnik version of the *Kauffman polynomial* are the unique Laurent polynomials $H_L, F_L \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ satisfying $H_{\text{unknot}} = F_{\text{unknot}} = 1$ and

$$\begin{aligned} xH_{L_+} - x^{-1}H_{L_-} &= yH_{L_0}, \\ xF_{L_+} - x^{-1}F_{L_-} &= y(F_{L_0} - x^{w(L_\infty) - w(L_0)}F_{L_\infty}), \end{aligned}$$

where L_+, L_-, L_0 and L_s are as in the definition of the Conway polynomial (cf. §3), L_∞ is obtained by changing the orientation of the "right" of the two loops in L_s (corresponding to either the two intersecting components or the two lobes of the singular component) and oriented smoothing of the crossing of the obtained singular link L_s', and $w(L)$ denotes the *writhe* of the diagram of L, i.e. the number of positive crossings minus the number of negative crossings (so that $w(L_+) - 1 = w(L_-) + 1 = w(L_0)$). The versions of H_L and F_L in [Lic] are obtained as $H_L(-ia, iz)$ and $(-1)^{m-1}F_L(-ia, iz)$.

Theorem 5.1. *Let e^t denote the (formal) power series $\sum \frac{t^n}{n!}$, and consider the power series*

$$H_L^* := \frac{H_L}{H_{K_1} \cdots H_{K_m}} \qquad \text{and} \qquad F_L^* := \frac{F_L}{F_{K_1} \cdots F_{K_m}},$$

where $K_1, \dots, K_m$ denote the components of the link L.

(i) For each $L\colon mS^1 \hookrightarrow S^3$ and any $n \in \mathbb{N}$ there exists an $\varepsilon_n > 0$ such that if $L'\colon mS^1 \hookrightarrow S^3$ is C^0 ε_n-close to L,

$$\begin{aligned} H_{L'}^*(e^{ch/2}, e^{h/2} - e^{-h/2}) &\equiv H_L^*(e^{ch/2}, e^{h/2} - e^{-h/2}) \mod (h^n), \\ F_{L'}^*(e^{(c-1)h/2}, e^{h/2} - e^{-h/2}) &\equiv F_L^*(e^{(c-1)h/2}, e^{h/2} - e^{-h/2}) \mod (h^n). \end{aligned}$$

(ii) H_L^* and F_L^* can be uniquely extended to all TOP links in S^3, preserving (i).
(iii) The extended H_L^* and F_L^* are invariant under TOP isotopy of L.

Proof. The connected sum formulae for H_L and F_L [Lic; Prop. 16.2] imply that H^* and F^* are invariant under PL isotopy. (Note that the connected sum in [Lic] is Hashizume's, not the componentwise connected sum in [MR1].) On the other hand, it was noticed in [Lie] (compare [Gu1], [BL], [St; Remark 4.7(1)]) that the coefficients of the power series

$$H_L(e^{ch/2}, e^{h/2} - e^{-h/2}) = \sum_{k=0}^{\infty} \sum_{i=0}^{k+m-1} p_{ki} c^i h^k \in \mathbb{Q}[c][[h]]$$
$$F_L(e^{(c-1)h/2}, e^{h/2} - e^{-h/2}) = \sum_{k=0}^{\infty} \sum_{i=0}^{k+m-1} q_{ki} c^i h^k \in \mathbb{Q}[c][[h]]$$

are (monochromatic) finite type invariants of L. Specifically, each p_{ki} and each q_{ki} is of type k, moreover $p_{0i} = q_{0i} = \delta_{m-1,i}$ (the Kronecker delta). (The argument in [Lie] was for $H_L H_{T_2}$ and $F_L F_{T_2}$, where T_2 denotes the trivial 2-component link, but it works as well for H_L and F_L, compare [Ba; proof of Theorem 3].) The rest of the proof repeats that of Theorem 3.2. □

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Steklov Mathematical Institute, Division of Geometry and Topology;
ГСП-1, ul. Gubkina 8, Moscow 117966, Russia

Current address: University of Florida, Department of Mathematics;
358 Little Hall, PO Box 118105, Gainesville, FL 32611-8105, U.S.

E-mail address: `melikhov@math.ufl.edu, melikhov@mi.ras.ru`